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BASS'S FIRST STABLE RANGE CONDITION

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0. Introduction

Let R be an associative ring with unit. Bass's lowest (the first) stable range condition [1] asserts the following: if a and b in R satisfy $Ra+Rb=R$, then there exists t in R with $a+tb$ left invertible (that is, $R(a+tb)=R$; Theorem 2.6 below says that the requirement that $a+tb$ be a unit is not really stronger). More exactly, this is the 'left' version of the condition, and there is a symmetric 'right' version; but the two versions are in fact equivalent (see Theorem 2.1 below).

For brevity, we call a ring satisfying the condition a B-ring. The expression 'stable range (or rank) of R is 1', or '$sr(R)\leq 1$', or 'a ring of stable range 1' is used for this in [19] and other places.

In this note known results are collected and a little information is added on B-rings. We do not introduce here Bass's higher stable range conditions (see [1, 3, 9, 11, 13, 14, 16–20, 22]) which follow from the first one by [19, Theorem 1], or non-associative B-rings (see [4, 5], where 'ring of stable range 2' means a ring satisfying the first stable range condition). A nice geometric characterization of B-rings is given in [21].

The note was inspired by (and called after) an old unpublished paper by I. Kaplansky which contains Theorems 2.4, 2.6, 2.8, and 5.3, so I dedicate it to him. I thank him for permition to publish his results and pointing out the paper [6] (which contains Theorems 2.2, 2.7, and the equivalence of 5.3(a) with 5.3(d); a ring has 'substitution property' in the sense of [6] if and only if it is a B-ring), R. Herman for discussions of the stable range of C^*-algebras, and a referee for suggestions.

1. Examples of B-rings

Example 1.1. Any field or a division ring is a B-ring. More generally, any Artinian ring (in particular, any finite-dimensional algebra over a field) is a B-ring (see Bass [1]; also it follows from Theorems 2.2 and 2.3 below).

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Example 1.2. The ring of all algebraic integers is a B-ring. More generally, if R is a commutative ring with 1 such that the multiplicative group of R/Rb is torsion for every non-zero b in R and the equation $t^n + ct^{n-1} + d = 0$ has a solution t in R for any natural number n and any c and d in R, then R is a B-ring. Indeed, let such a ring R be given, and let a and b in R satisfy $Ra + Rb = R$. If $b = 0$, then $R(a + tb) = Ra = Ra + Rb = R$ for $t = 0$. Otherwise, we find a natural number n and an element z in R such that $(-a)^n - 1 = zb$. Since $Ra^{n-1} + Rb^{n-1} = R$, we can find c and d in R such that $-z = c(-a)^{n-1} + db^{n-1}$. Pick t in R satisfying $t^n + ct^{n-1} + d = 0$. Then

$$(a + tb - a)^n + bc(a + tb - a)^{n-1} + b^n d = 0,$$

hence

$$R(a + tb) \ni (-a)^n + (-a)^{n-1}bc + b^n d = 1 + b(z + (-a)^{n-1}c + b^{n-1}d) = 1,$$

so $R(a + tb) = R$.

Example 1.3 (Rubel [15]). The ring of all entire functions (of one complex variable) is a B-ring.

Example 1.4 (see [10]). The disc algebra (consisting of functions analytic on the unit disc and continuous on its closure) is a B-ring.

Example 1.5 (Estes–Ohm [3]). The Kronecker function ring of any integrally closed domain is a B-ring.

Example 1.6 (see [3]). A finitely generated commutative algebra R over a field is a B-ring if and only if $\dim(R) \leq 0$.

Example 1.7 (Vaserstein [19]). The ring of all continuous real (complex, quaternion) functions on a topological space X is a B-ring if and only if the dimension of X is at most 0 (respectively, 1, 3). The same is true for dense subrings of continuous functions.

The next sections provide more examples of B-rings.

2. General properties of B-rings

First we note that the notion of B-ring is in fact left-right symmetric:

Theorem 2.1 (see [19]). *A ring R is a B-ring if and only if so is the opposite ring R_{op}.*

Secondly, in studying B-rings we might assume them semi-simple, for it is a triviality that if I is an ideal of a ring R contained in its Jacobson radical rad(R), then R is a B-ring if and only if so is R/I:

Theorem 2.2. *A ring R is a B-ring if and only if so is $R/\mathrm{rad}(R)$.*

The next result is also trivial:

Theorem 2.3. *If R is the direct product of a family $\{R_\alpha\}$ of rings, then R is a B-ring if and only if so is each R_α.*

In the next theorem M_nR denotes the ring of all n by n matrices over R.

Theorem 2.4 (see [19]). *For any natural number n, a ring R is a B-ring if and only if so is M_nR.*

Theorem 2.5 (see [1] or [19]). *Any factor ring of a B-ring is a B-ring.*

The generalizations to rings of higher stable range of the following 3 theorems are false. Namely, there is a ring satisfying the second stable range condition containing a left unit, which is not a right unit, and an idempotent p such that pRp does not satisfy any stable range condition.

Theorem 2.6. *In a B-ring, one-sided inverses are two-sided.*

Proof. Let $ax=1$. For $b=1-xa$ we have $Ra+Rb=R$. Hence there exists t with $u=a+tb$ left invertible. Since $bx=x-xax=x-x=0$, $1=ux$ so that u is also right invertible. Thus, u is a unit, and so are x and a.

The following theorem generalizes Theorem 2.6.

Theorem 2.7. *Let R be a B-ring, M and M' (right) R-modules, and P a finitely generated projective R-module. Then any isomorphism $M\oplus P\sim M'\oplus P$ induces an isomorphism $M\sim M'$.*

Proof. Since P is a direct summand in R^n for some n, it suffices to prove the cancellation for the free R-modules R^n. Proceeding by induction on n, it is enough to consider the case $n=1$. The cancellation for such $P=R$ is equivalent to the statement that the automorphism group of the R-module $M\oplus R$ acts transitively on the unimodular elements v. Recall that v is unimodular if $fv=1$ for some R-module morphism $f: M\oplus R\to R$. Given such v and f, we can write $v=\binom{r}{m}$ and $f=(s, g)$ with r and s in R, m in M, $g: M\to R$, and $fv=sr+gm=1$. Since $Rr+Rgm=R$ and R is a B-ring, there are t and x in R such that $x(r+tgm)=1$. Then

$$\begin{pmatrix} x & 0 \\ -mx & 1 \end{pmatrix}\begin{pmatrix} 1 & tg \\ 0 & 1 \end{pmatrix} v = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

(Here 1 is used also for id_M.) Thus, every unimodular v is taken to $\binom{1}{0}$ by an automorphism of $M \oplus R$.

Theorem 2.8. *If R is a B-ring and $p^2 = p \in R$, then pRp is also a B-ring.*

Proof. Let a and b be in $pRp = R'$ and $R'a + R'b = R'$. Consider $a + 1 - p$ and b in R. We have $R'(1-p) = 0$, so $R(a+1-p) + Rb \supset R'a + R'b \ni p$. On the other hand, $(1-p)a = 0 = (1-p)b$. So

$$R(a+1-p) + Rb \ni (1-p)(a+1-p) + (1-p)b = 1-p.$$

Thus, $R(a+1-p) + Rb \ni p + 1 - p = 1$.

Since R is a B-ring, there is t in R such that $R(a+tb+1-p) = R$. We have

$$(1-(1-p)tb)(1+(1-p)tb) = 1 = (1+(1-p)tb)(1-(1-p)tb),$$

so $1-(1-p)tb$ is a unit of R, hence

$$R = R(a+tb+1-p)(1-(1-p)tb) = R(a+ptb+1-p).$$

Therefore $R'(a+ptpb) = R'$.

Theorems 2.1, 2.4, and 2.8 imply:

Corollary 2.9. *Let R be a B-ring. Then the endomorphism ring of any finitely generated right or left projective R-module is a B-ring.*

Remark. To obtain Theorem 2.6 from Theorem 2.7, note that $xa = 1$ implies that $R = aR \oplus (1-ax)R$ with $aR = axR \sim R$, hence $1 - ax = 0$.

Remark. Corollary 2.9 shows that the notion of B-ring is Morita invariant.

3. Rings without units

Definition 3.1. For any associative ring R, let R_1 denote the ring of all pairs (r, z), where $r \in R$ and $z \in Z$(the integers), with addition and multiplication given by

$$(r, z) = (r', z') = (r + r', z + z')$$

and

$$(r, z)(r', z') = (rr' + zr' + rz', zz').$$

Then R_1 is an associative ring with unit. We will identify R with the two-sided ideal $(R, 0)$ of R_1 and Z with the subring $(0, Z)$ of R_1. Note that $R_1/R = Z$.

Definition 3.2. A ring R is called a B_1-ring (resp., B'-ring), if for any a and b in R_1 with $a-1$ in R (respectively, $b-1$ in R) and $R_1a+R_1b=R_1$ there is t in R_1 such that $R_1(a+tb)=R_1$.

Theorem 3.3. *Every B_1-ring is a B'-ring.*

Proof. Let R be a B_1-ring, $a, b\in R_1, b-1\in R$, and $R_1a+R_1b=R_1$. Then $R_1(a+(1-a)b)+R_1b=R_1$. Since $a+(1-a)b-1\in R$ and R is a B_1-ring, there is t in R_1 such that

$$R_1(a+(1-a)b+tb)=R_1,$$

hence $R_1(a+t'b)=R_1$ for $t'=1-a+t\in R_1$.

Theorem 3.4. *A ring R with unit is a B-ring if and only if it is a B_1-ring (or B'-ring).*

Proof. Let 1_R be the identity element of R. Define the ring morphism $F: R_1\to R$ by $F(r, z)=z1_R+r$. (By the way, the ring R_1 is the direct product of $R=FR_1$ and $Z(1-1_R)$.)

Assume first that R is a B-ring, and let $a, b\in R_1, a-1\in R$, and $R_1a+R_1b=R_1$. Then $\binom{Fa}{Fb}\in R^2$ is unimodular, so $R(Fa+tFb)=R$ for some t in R.

Then $R_1(a+tb)\supset R(a+tb)=R\ni a+tb-1$. On the other hand, $R_1(a+tb)\ni a+tb$. Therefore

$$R_1(a+tb)\ni a+tb-(a+tb-1)=1.$$

Thus, R is a B_1-ring and, by Theorem 3.3, a B'-ring.

Assume now that R is a B'-ring, and let $a, b\in R$ and $Ra+Rb=R$. We write $xa+yb=1_R$ with x and y in R. Then

$$xa+(y-1_R+1)(b-1_R+1)=1.$$

Since R is a B'-ring, there is t in R such that

$$R_1(a+t(b-1_R+1))=R_1.$$

Applying F, we obtain that $R(a+(Ft)b)=R$, where $Ft\in R$.

Thus, R is a B-ring.

Lemma 3.5. *Let a ring R be contained as a right ideal in a ring R' with unit $1'$ (for example, $R'=R_1$), $a, b\in R'$, and $a-1'\in R$. Then the following 4 conditions are equivalent:*

(a) $R'a+R'b=R'$,

(b) $R'a+R'b\ni 1'$,

(c) $Ra+Rb=R$,
(d) $(1'+R)a+Rb=1'+R$.

Proof. It is clear that (a) and (b) are equivalent.

Let us prove that (b) implies (c). For an arbitrary c in R, we have

$$Ra+Rb\supset cR'a+cR'b\ni c$$

(using (b) and $R\supset RR'\supset cR'$).

Let us prove that (c) implies (d). For an arbitrary c in R we have

$$(1'+R)a+Rb=a+Ra+Rb=a+R=1'+R$$

(using (c) and $a-1'\in R$).

Since R' contains both R and $1'+R$, it is clear that (d) implies (b).

Theorem 3.6. *Let R be an associative ring, n a natural number, and R' as in Lemma 3.5. Then the following conditions are equivalent:*

(a) *R is a B_1-ring.*

(b) *For any a and b in R_1 such that $a-1, b\in R$ and $R_1a+R_1b=R_1$ there is t in R such that $R_1(a+tb)=R_1$.*

(c) *R_{op} is a B_1-ring.*

(d) $R/\mathrm{rad}(R)$ is a B_1-ring.

(e) *M_nR is a B_1-ring.*

(f) *Every factor ring of R is a B_1-ring.*

(g) *Every right and every left ideal of R is a B_1-ring.*

(h) *Under the conditions of Lemma* 3.5 (*when* 3.5 (a–d) *hold*) *there is t in R such that $R'(a+tb)=R'$.*

Proof. Assume first (a), and let us prove (h). Let a and b be as in 3.5. By Lemma 3.5, $xa+yb=1'$ for some x in $1'+R$ and y in R. Set

$$a_1:=(a-1')+1\in R+1\subset R_1,$$

$$x_1:=x-1'+1\in R_1,\qquad b_1:=yb\in R.$$

Then $x_1a_1+b_1=1$, hence $R_1a_1+R_1b_1=R_1$, and $a_1-1\in R$. By (a), there are t_1 and s in R_1 such that $s(a_1+t_1b_1)=1$. Let $F:R_1\to R'$ be the ring homomorphism identical on R and taking 1 to $1'$. Then $F(t_1b_1)=t_1b_1$ (since $R_1R\subset R$ and $RR'\subset R$), $Fa_1=a$, and $(Fs)(a+t_1b_1)=1'$, hence $R'(a+t_1yb)=R'$. Since $t:=t_1y\in R_1R\subset R$, we are done.

Let us prove now that (h) implies (b). Let a and b be as in (b) (in fact, instead of $b\in R$ we are going to use the weaker condition $b\in R_1$). By Lemma 3.5, $xa+yb=1$ with $x\in 1+R$ and $y\in R$. Set

$$a':=(a-1)+1' \quad\text{and}\quad x':=(x-1)+1'\in R'+1'.$$

Then $x'a'+yb=1'$ and $yb\in R$. By (h), there are $t\in R$ and $s\in R'$ such that $s(a'+tyb)=1'$. Set

$$s_1:=(1'-a'-tyb)s+1\in RR'+1\subset R_1.$$

Then $s_1(a+tyb)=1$, so $R_1(a+tyb)=R_1$.

Now let us show that (b) implies (a). Let a and b be as in Definition 3.2 of B_1-rings. By Lemma 3.5, $xa+yb=1$ for some x in $1+R$ and y in R. By (b), $R_1(a+t'yb)=R_1$ for some t' in R, hence

$$R_1(a+tb)=R_1 \quad \text{for } t=t'y'\in R\subset R_1.$$

Thus, the conditions (a) and (b) are equivalent. The condition (b) means that $\mathrm{sr}(R)\leq 1$ in the sense of [19]. It is contained in results of [19] that (b) is equivalent with (c), (d), (e), (f), and (g) with two-sided ideals. So it leaves to us only to prove that any left ideal J of a B_1-ring R is a B_1-ring.

Let $a, b, x, y\in J_1$, $a-1\in J$, and $xa+yb=1$. Set

$$x'=1+(1-a)x\in 1+J \quad \text{and} \quad y'=(1-a)y\in J.$$

Then $x'a+y'b=1$. Since R is a B_1-ring, $s(a+t'y'b)=1$ for some s and t' in R_1. Since $t=t'y'\in J$, $a+tb-1\in J$. Since $s(a+tb)=1$, $s-1\in J$, so $s\in J_1$.

The following theorem shows, in particular, that not every B'-ring is a B_1-ring.

Theorem 3.7. *Let R be an associative ring and X an infinite set. Denote by M_XR the ring of all matrices $(a_{i,j})_{i,j\in X}$ with only finitely many non-zero entries in each matrix. Then:*

(a) *M_XR is a B_1-ring if and only if so is R.*

(b) *M_XR is always a B'-ring.*

Proof. (a) If R is a B_1-ring, so is M_nR for any n (see Theorem 3.4). Since every finite subset of M_XR is contained in a subring isomorphic to M_nR for some n, it follows that M_XR is a B_1-ring.

Assume now that M_XR is a B_1-ring. Pick x in X. Given any a and b in R_1 with $a-1\in R$ and $R_1a+R_1b=R_1$, we define matrices $a'=(a'_{i,j})$ and $b'=(b'_{i,j})$ in $(M_XR)_1$ by

$$a'_{x,x}=a, \qquad a'_{i,i}=1 \quad \text{for } i\neq x, \quad \text{and} \quad a'_{i,j}=0 \quad \text{for } i\neq j;$$

$$b'_{x,x}=b \quad \text{and} \quad b'_{i,j}=0 \quad \text{for } (i,j)\neq(x,x).$$

Then $(M_XR)_1a'+(M_XR)b'=(M_XR)_1$, so there is $t=(t_{i,j})$ in $(M_XR)_1$ such that $(M_XR)_1(a'+t'b')=(M_XR)_1$.

Writing matrices in block form according to the decomposition $X=\{x\}\cup\{X-x\}$, we have:

$$a'+t'b'=\begin{pmatrix} a+tb & 0 \\ v & 1 \end{pmatrix},$$

where (t/v) is the x-th column of t'.

Since the matrix $\begin{pmatrix}1&0\\v&1\end{pmatrix}$ is invertible, it follows that $\begin{pmatrix}a+tb&0\\0&1\end{pmatrix}$ is left invertible, hence $a+tb$ is left invertible in R_1. (We use 1 to denote different identity matrices.) Thus, R is a B_1-ring.

(b) Let a, b, x, y be in $(M_XR)_1$, $b-1\in M_XR$, and $xa+yb=1$. Find a finite subset S of X such that for $(c_{i,j})$ equal to any of the matrices a, b, c, d we have: $c_{i,j}=0$ when i or j is outside of S and $i\neq j$, and $c_{i,i}=c_{j,j}$ when both i and j are outside of S (we write elements of $Z\subset(M_XR)_1$ as scalar matrices). Writing

$$a=\begin{pmatrix}a'&0\\0&z\end{pmatrix}\quad\text{and}\quad b=\begin{pmatrix}b'&0\\0&1\end{pmatrix}$$

according to $X=S\cup(X-S)$, we obtain that $x'a'+y'b'=1$, where $a', b', x', y'\in(M_SR)_1$ and $b-1\in M_SR$. Pick disjoint S' and S'' in $X-S$ of the same finite cardinality as S. We will write now matrices in $(M_XR)_1$ in block form as 4 by 4 matrices according to

$$X=S\cup S'\cup S''\cup(X-S-S'-S'').$$

Set

$$t=\begin{pmatrix}0&0&1&0\\1&1-z&0&0\\0&-y'&-x'-z&0\\0&0&0&1-z\end{pmatrix}\in(M_XR)_1.$$

Then

$$a+tb=\begin{pmatrix}a'&0&1&0\\b'&1&0&0\\0&-y'&-x'&0\\0&0&0&1\end{pmatrix}$$

is invertible, because the finite matrix

$$\begin{pmatrix}a'&0&1\\b'&1&0\\0&-y'&-x'\end{pmatrix}=\begin{pmatrix}1&0&0\\0&1&0\\-x'&-y'&1\end{pmatrix}\begin{pmatrix}1&0&a'\\0&1&b'\\0&0&1\end{pmatrix}\begin{pmatrix}0&0&1\\0&1&0\\1&0&0\end{pmatrix}$$

is invertible.

Example 3.8. If we take a ring R with one-sided units which are not two-sided, then M_XR will be a B'-ring with:

(a) one-sided units in $(M_XR)_1$ which are not two-sided;

(b) idempotents p such that pBp is not a B-ring;

(c) left and right ideals which are not B'-rings.

In particular, one cannot replace B_1 by B' in the following result.

Theorem 3.9. *If R is a B_1-ring, then every one-sided unit in R_1 is two-sided, and pRp is a B-ring for any idempotent $p = p^2$ in R.*

Proof repeats the proofs of Theorems 2.6 and 2.8 above. We leave it to the reader, as well as proving the following result.

Theorem 3.10. *Let R be an associative ring and n a natural number. Then the following statements are equivalent:*

(a) *R is a B'-ring.*

(b) *For any a and b in R_1 such that $a, b-1 \in R$ and $R_1a + R_1b = R_1$ there is t in R_1 such that $R_1(a+tb) = R_1$.*

(c) *R_{op} is a B'-ring.*

(d) *$R/\mathrm{rad}(R)$ is a B'-ring.*

(e) *M_nR is a B'-ring.*

(g) *For every ring R' with unit $1'$ containing R as a two-sided ideal and any $a, b \in R'$ with $b - 1' \in R$ and $R'a + R'b = R'$ there is $t \in R'$ such that $R'(a+tb) = R'$.*

Remark. One-sided ideals of rings satisfying the second stable range condition (see [1]) need not satisfy any stable range condition. For example, in the conditions of Theorem 3.7, both M_XR and $(M_XR)_1$ always satisfy the second stable range condition, and they have one-sided ideals J such that $J/\mathrm{rad}(J)$ is isomorphic to R (which could be of infinite stable range).

4. C^*-algebras

For any C^*-algebra R, let $R+C$ denote the C^*-algebra with unit obtained from R by adjoining an identity element (so $(R+C)/R = C$, the complex numbers). The following result is evident.

Theorem 4.1. *A C^*-algebra R is a B_1-ring if and only if $R+C$ is a B-ring.*

Theorem 4.2. *Let R be a C^*-algebra with unit. Then the following conditions are equivalent:*

(a) *R is a B-ring.*

(b) *For any a and b in R with $Ra + Rb = R$ there is an unitary $t = t^{*-1}$ in R such that $R(a+tb) = R$.*

(c) *For any a and b in R with $Ra + Rb = R$ and any positive* real *number ε there is t in R such that $R(a+tb) = R$ and $\|t\| < \varepsilon$.*

(d) *The invertible elements are dense in R.*

(e) *$R \otimes K$ is a B_1-ring, where K is the algebra of all compact operators on a separable infinite-dimensional complex Hilbert space.*

Proof. By [9], (a) implies (d) (the commutative case was done in [19]). This together with results mentioned and obtained in Rieffel [13] gives the theorem.

In the commutative case (which is covered by Example 1.7) one obtains a more complete result:

Theorem 4.3. *A commutative C^*-algebra R is a B_1-ring, if and only if the space of maximal ideals of $R+C$ has dimension ≤ 1.*

Example 4.4 (see [13]). Any AF C^*-algebra is a B_1-ring.

Example 4.5 (see [13]). If G is a compact group, then its C^*-algebra $C^*(G)$ is a B-ring.

Example 4.6 (Handelman [8]). Suppose R is a C^*-algebra such that for every a in R there is a unitary u with au positive. Then R is a B-ring.

Example 4.7 (Riedel [12]). Some (perhaps, all the) irrational rotation algebras are B-ring.

Example 4.8 (see [2]). Taking $R=C$ in Theorem 4.2, we see that $R\otimes K=K$ is a B_1-ring. Note that K is a two-sided ideal in the ring of bounded operators which do not satisfy any stable range condition.

5. Von Neumann regular rings

Definition 5.1. An associative ring R with 1 is called *von Neumann regular,* if for any a in R there is x in R such that $axa=a$.

Definition 5.2. Two idempotents e and f in a ring R are *equivalent* if there are x in eRf and y in fRe with $xy=e$ and $yx=f$.

Theorem 5.3. *For any von Neumann regular ring R the following conditions are equivalent:*

(a) *R is a B-ring.*
(b) *For any a in R there is a unit u in R such that $aua=a$.*
(c) *The conclusion of Theorem 2.7 holds.*
(d) *The conclusion of Theorem 2.7 holds when $M\oplus P=R$.*
(e) *If e and f are equivalent idempotents in R, then so are $1-e$ and $1-f$.*

Proof. Theorem 4.12 of Goodearl [7] says that (a) and (b) are equivalent. By Theorem 2.7, (a) implies (c). Evidently, (c) implies (d). Let us show that (d) implies (e), and that (e) implies (b).

Assume first (d), and let e and f be as in (e). Let x and y be as in Definition 5.2. Then $r \mapsto yr$ and $r' \mapsto xr'$ give mutually inverse R-module homomorphisms $eR \to fR$ and $fR \to eR$, so $eR \sim fR$. Using again that $e^2 = e$ and $f^2 = f$, we see that $R = eR \oplus (1-e)R = fR \oplus (1-f)R$. By (d), $(1-e)R \sim (1-f)R$.

Pick mutually inverse isomorphisms $(1-e)R \to (1-f)R$ and $(1-f)R \to (1-e)R$, and extend them by the zero mappings (on the complementary summands) to endomorphisms $r \mapsto sr$ and $r \mapsto tr$ respectively of the R-module R. Then $se = fs = 0 = tf = et$ and $ts(1-e) = 1-e$, $st(1-f) = 1-f$, hence $ts = 1-e$, $st = 1-f$ and $t = (1-e)t(1-f)$, $s = (1-fs(1-e)$. Thus, $1-e$ and $1-f$ are equivalent.

Assume now (e). We want to prove (b). Let $a \in R$. Since R is regular, $axa = a$ for some x in R. Then ax and xa are idempotents, so by (e) there are s in $(1-ax)R(1-xa)$ and t in $(1-xa)R(1-ax)$ such that $st = 1-ax$ and $ts = 1-xa$.

Then $(a+s)(xax+t) = 1 = (xax+t)(a+s)$, so $u = xax + t$ is a unit. Moreover, $aua = axaxa + ata = a + 0 = a$.

Thus, (e) implies (b), and the theorem is proved.

Example 5.4. If a (von Neumann) regular ring R with 1 is commutative *or*, more generally, abelian (that is, every idempotent in R is central), then R is a B-ring (see [7, Corollary 4.5]). Here is a proof of 5.3(b) for any abelian regular R with 1. Let a be in R. By the regularity, $axa = a$ for some x in R. Then $xaxa = xa$ is an idempotent. Since R is abelian, xa commutes with both a and x. Set $u := xax + 1 - xa$. Then $aua = a$ and

$$u(xax + 1 - xa) = (xax + 1 - xa)u = 1,$$

so u is a unit.

But [11] there are (non-abelian) regular rings with 1 which are not B-rings (that is, which do not satisfy the first stable range condition; some of them satisfy the second stable range condition which was mentioned but not introduced in this note).

Remarks added in proof

(a) Example 1.4 is mentioned in a paper by G. Corach and A.R. Larotonda in this journal, Vol. 32, No. 3 (1984), pp. 289–300. A more general result was obtained in "Stable range in holomorphic function algebras" by G. Corach and F.D. Suárez.

(b) For more information about von Neumann regular B-rings (known as unit-regular rings) see "Reviews in Ring Theory" by Small.

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